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Multiparameter Bifurcation Problems and Topological Degree*

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1. INTRODUCTION

Consider the operator equation

$$u = F(\lambda, u), \tag{1.1}$$

where $F: \mathbb{R}^k \times E^k \to E^k$ is a completely continuous mapping, $k \geqslant 1$, and E is a real Banach space. Assume that F has the form

$$F(\lambda, u) = A(\lambda)u + H(\lambda, u), \tag{1.2}$$

where

$$A(\lambda) = \sum_{i=1}^{k} \lambda_i \begin{bmatrix} A_{11}^{(i)} & \cdots & A_{1k}^{(i)} \\ \vdots & & \vdots \\ A_{k1}^{(i)} & \cdots & A_{kk}^{(i)} \end{bmatrix}. \tag{1.3}$$

Here $u = [u_1,..., u_k]^{\mathrm{T}}$, $\lambda = (\lambda_1, \lambda_2,..., \lambda_k)$, $A_{mn}^{(i)}: E \to E$ is a compact linear operator, and $H(\lambda, u)/\|u\| \to 0$ uniformly for λ contained in compact subsets of $\mathbb{R}^k$.

If $k = 1$, (1.1)–(1.3) is an operator equation widely studied in bifurcation theory (see, e.g., work of Crandall and Rabinowitz [6], Krasnosel'skii [9], Rabinowitz [13, 14], Schmitt and Smith [15], and Turner [18]) via the topological degree of Leray and Schauder. The case $k > 1$ has recently attracted considerable mathematical attention. For example, equations of or similar to the form (1.1)–(1.3) have appeared in connection with multiparameter generalizations of nonlinear Sturm–Liouville boundary value problems in papers of Browne and Sleeman [2, 3] and in connection with

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three point boundary value problems in second-order ordinary differential equations in a recent paper of Hale [7].

The purpose of this paper is to examine Eqs. (1.1)–(1.3) for $k > 1$ by means of the Leray–Schauder topological degree. To this end, in Section 2, we establish an analogue to the Krasnosel'skii bifurcation theorem [9] for higher dimensional parameter spaces and note some of the topological consequences of this theorem in the case $k > 1$. Two global results also appear in Section 2. The first of these results is a multiparameter version of the Rabinowitz bifurcation theorem [14]. The second establishes three global possibilities in case the set of bifurcation points in parameter space contains a compact connected subset which separates $\mathbb{R}^k$ into two components. Section 3 examines a strongly coupled system of Sturm–Liouville boundary value problems which illustrates this three alternative theorem. Finally, in Section 4, we consider a system which has its origin in a study of axisymmetric buckling of spheres made by Bauer, Reiss, and Keller [1].

2. Local and Global Bifurcation Results

Consider the operator equations (1.1)–(1.3). We will call a point $(\lambda_0, 0) \in \mathbb{R}^k \times E^k$ a *bifurcation point* for (1.1)–(1.3) provided that every open set in $\mathbb{R}^k \times E^k$ containing $(\lambda_0, 0)$ also contains a point (λ, u), with $u \neq 0$, such that (λ, u) satisfies (1.1)–(1.3). Let $\mathscr{B}$ and Σ_A denote, respectively, the sets $\{\lambda \in \mathbb{R}^k: (\lambda, 0)$ is a bifurcation point for (1.1)–(1.3)$\}$ and $\{\lambda \in \mathbb{R}^k: \lambda = (\lambda_1,..., \lambda_k)$ is such that the null space $N(I - A(\lambda)) \neq \{0\}\}$. Then $\mathscr{B}$ and Σ_A are closed in $\mathbb{R}^k$, and $\mathscr{B} \subseteq \Sigma_A$.

This last statement gives a necessary condition for bifurcation to occur in the setting of (1.1)–(1.3). We now present a result which establishes a sufficient condition in this respect. First, observe that if $\lambda \in \Sigma_A$, then the algebraic multiplicity of λ, denoted mult λ, is the dimension of the subspace $\bigcup_{j=1}^{\infty} N\{(I - A(\lambda))^j\}$ of E^k.

Theorem 2.1. *Let $\lambda_0 \in \Sigma_A$ have odd algebraic multiplicity. Then $(\lambda_0, 0)$ is a bifurcation point for* (1.1)–(1.3).

Proof. Observe that (1.3) guarantees a change in the topological index (see [10]) of $I - A(\lambda)$ at λ_0 through the ray emanating from 0. The result may then be verified using the topological degree of Leray–Schauder [10] in a manner similar to that of [9].

Theorem 2.1 has several immediate consequences.

Corollary 2.2. *Suppose λ_0 is as in the statement of Theorem* 2.1. *Then $\mathbb{R}^k \backslash \Sigma_A$ is not connected.*

Proof. There exists $\lambda_1, \lambda_2 \in \mathbb{R}^k \backslash \Sigma_A$ such that $0, \lambda_0, \lambda_1, \lambda_2$ are collinear and λ_0 is the only element of Σ_A on the line segment joining λ_1 and λ_2. If $\mathbb{R}^k \backslash \Sigma_A$ is not connected, since Σ_A is closed in $\mathbb{R}^k$, $\mathbb{R}^k \backslash \Sigma_A$ is path connected. There is then a path h: $[0, 1] \to \mathbb{R}^k \backslash \Sigma_A$ such that $h(0) = \lambda_1$ and $h(1) = \lambda_2$. Define $M(t)$ by $M(t)x = x - A(h(t))x$ for $(x, t) \in E^k \times [0, 1]$. Then, since $M(t)x = 0$ only if $x = 0$, the homotopy property of the Leray–Schauder degree implies that $\deg_{\mathrm{LS}}(I - A(\lambda_1), B(0, 1), 0) = \deg_{\mathrm{LS}}(I - A(\lambda_2), B(0, 1), 0)$, where $B(0, 1)$ is the unit ball in E^k, a contradiction.

Corollary 2.2 shows that if $\lambda \in \Sigma_A$ has odd algebraic multiplicity, then Σ_A has codimension 1 in $\mathbb{R}^k$. The next result indicates that often a surface of bifurcation points exists about a point $\lambda \in \Sigma_A$ with mult λ an odd integer.

Corollary 2.3. *If $\lambda_0 \in \Sigma_A$ is as in Theorem* 2.1 *and Σ_A is a $k - 1$ nonsingular manifold T at λ_0 then $T \subseteq \mathscr{B}$.*

Proof. The hypotheses guarantee the existence of $\delta > 0$ such that, for all $\lambda \in T$ sufficiently close to λ_0, $(1 \pm \delta)\lambda \in \mathbb{R}^k \backslash \Sigma_A$ and λ is the only element of Σ_A on the line segment connecting $(1 - \delta)\lambda$ and $(1 + \delta)\lambda$. The result then follows from the homotopy invariance of the Leray–Schauder topological degree.

The preceding results are obtained in situations other than (1.1)–(1.3). For example, if, in (1.3), operator $A_{mn}^{(i)}$ is the zero operator for $m \neq n$ and $H(\lambda, u) = (H_1(\lambda, u_1),..., H_k(\lambda, u_k))$, where $u = (u_1,..., u_k)$, then the real Banach space E^k may be replaced by $E_1 \times \cdots \times E_k$, where E_i, $i = 1, 2,..., k$, is a real Banach space and E_i and E_j are not necessarily the same space if $i \neq j$, $i, j = 1, 2,..., k$. Such situations arise in the study of systems of ordinary differential equations. In fact, in [4], we present a nonlinear boundary value problem based on Klein's oscillation theorem whose solution makes use of Theorem 2.1 in this setting.

Let $\lambda_0 \in \Sigma_A$. Suppose $h: \mathbb{R} \to \mathbb{R}^k$ is a line such that $h(0) = \lambda_0$. A unit vector $\mathbf{u}_h$ in the direction of h (we assume that if $\tau \in \mathbb{R}$ is such that $\|h(\tau)\| = \min_{t \in \mathbb{R}} \|h(t)\|$, then $\tau < 0$) is a *direction of changing degree at* λ if the following conditions hold:

(i) there is a number $\varepsilon_h > 0$ such that $\deg_{\mathrm{LS}}(M(t), B(0, 1), 0)$ is defined for all t such that $|t| < \varepsilon_h$ and $t \neq 0$.

(ii) $\deg_{\mathrm{LS}}(M(\tau), B(0, 1), 0) = \operatorname{sgn}(\tau\beta) \cdot \deg_{\mathrm{LS}}(M(\beta), B(0, 1), 0)$ for all $\tau, \beta \in (-\varepsilon_h, \varepsilon_h)$, $\tau \neq 0$, $\beta \neq 0$.

Let $\mathscr{S}$ denote the closure in $\mathbb{R}^k \times E^k$ of the set $\{(\lambda, u) \in \mathbb{R}^k \times E^k: (\lambda, u)$ is a solution of (1.1)–(1.3) and $u \neq 0\}$. Let a continuum in $\mathscr{S}$ denote a closed connected subset. We then have

THEOREM 2.4. *Consider* (1.1)–(1.3). *Suppose* $\lambda_0 \in \Sigma_A$ *and* $\mathbf{u}_h$ *is a direction of changing degree at* λ_0. *Then there exists a subcontinuum* $\mathscr{C}$ *of* $\mathscr{S}$ *meeting* $(\lambda_0, 0)$ *such that either*

(i) $\mathscr{C}$ *is unbounded, or*

(ii) $\mathscr{C} \cap ([\mathscr{B} \times \{0\}] \setminus \{\lambda_0, 0)\}) \neq \varnothing$.

Proof. Let $t \in \mathbb{R}$ and $u \in E^k$. Define operators $G: \mathbb{R} \times E^k \to E^k$ and $\tilde{A}(t)$: $E^k \to E^k$ by

$$G(t, u) = F(h(t), u)$$

and

$$\tilde{A}(t)u = A(h(t))u.$$

Since F and A are completely continuous and compact linear, respectively, the same is true for G and $\tilde{A}(t)$. Since $\tilde{A}(t)$ is the linearization of $G(t, \cdot)$ and $\dim \bigcup_{r>1} N\{(I - \tilde{A}(0))^r\}$ is odd the proof of the Rabinowitz bifurcation theorem [14, Theorem 1.3] can be utilized in this situation. Thus if $\mathscr{S}$ denotes the closure in $\mathbb{R} \times E^k$ of $\{(t, u) \in \mathbb{R} \times E^k: u = G(t, u), u \neq 0\}$, there is a maximal subcontinuum of $\mathscr{S}$ containing $(0, 0)$ which is either unbounded or meets $(t_0, 0)$, $t_0 \neq 0$.

COROLLARY 2.5. *Let* $\mathscr{C}$ *denote the maximal continuum of* $\mathscr{S}$ *meeting* $(\lambda_0, 0)$, *where* λ_0 *is as in Theorem* 2.4. *Then for each direction* $\mathbf{u}_h$ *of changing degree at* λ *there is a subcontinuum* $\mathscr{C}_h$ *of* $\mathscr{C}$ *such that*

(i) $\mathscr{C}_h$ *satisfies the alternatives of Theorem* 2.4, *and*

(ii) *the projection of* $\mathscr{C}_h$, $\pi(\mathscr{C}_h)$ *into* $\mathbb{R}^k$ *is contained in* $h(\mathbb{R})$.

Remark 2.6. While Corollary 2.5 is a global result, it also gives information about the local behavior of solutions emanating from the bifurcation point $(\lambda_0, 0)$. For example, suppose there is a direction $\mathbf{u}_h$ of changing degree at λ_0 such $\pi(\mathscr{C}_h) \neq \{\lambda_0\}$. Then we may assert that $\pi(\mathscr{C}_{h'}) \neq \{\lambda_0\}$ for directions $\mathbf{u}_{h'}$ of changing degree at λ_0 in a cone about h. This fact is a consequence of a readily established extension of Theorem 2.4 to more general types of paths. In particular, if we view the problem $\mathbb{R}^k \times \bar{\mathbb{R}}^+$ (allowing $\bar{\mathbb{R}}^+$ to represent the norm of an element in E^k), we find that k-dimensional "surfaces" of nontrivial solutions appear. Thus the bifurcation phenomena associated with such situations are locally "higher dimensional."

Theorem 2.4 adapts the Rabinowitz bifurcation theorem to the situation of (1.1)–(1.3). In a sense, it also characterizes the structure of the nontrivial solutions to (1.1)–(1.3) emanating from a single bifurcation point of odd multiplicity. However, in the case of higher dimensional parameter spaces, one often has "surfaces" (or collections of "surfaces") of bifurcation points in parameter space. (These surfaces, as illustrated in [5], exhibit a wide variety of structural possibilities.) A natural question arises: can one give a description of the continuum of nontrivial solutions to (1.1)–(1.3) which emanate from a surface of bifurcation points? The next theorem is intended as a partial answer to this question.

THEOREM 2.7. *Consider* (1.1)–(1.3). *Let* $Z \subseteq \Sigma_A$ *have the following properties*:

(a) mult λ *is odd for all* $\lambda \in Z$,

(b) Z *is compact and connected,*

(c) $\mathbb{R}^k \setminus Z = \Omega \cup \Phi$, *where* Ω *and* Φ *are open disjoint connected sets with* Ω *bounded and* $Z = \partial\Omega$,

(d) *if* V *is any open subset of* Ω *such that* $\bar{V} \subseteq \Omega$ *then there is a point* $x \in V$ *such that* x *can be joined to* Z *by a line segment entirely contained in* Ω,

(e) *there is a neighborhood* Γ *of* Z *in* $\mathbb{R}^k$ *such that* $\bar{\Gamma} \cap \Sigma_A = Z$, *in particular,* $\partial\Gamma \cap \Sigma_A = \varnothing$.

Then if $\mathscr{C} = \{C: C$ *is a subcontinuum of* S *meeting* $Z \times \{0\}\}$, *one of the following holds*:

(i) $K = \bigcup_{C \in \mathscr{C}} C$ *is unbounded.*

(ii) *There is* $C \in \mathscr{C}$ *such that* C *meets* $(\mathbb{R}^k \setminus Z) \times \{0\}$.

(iii) *For all* $\lambda \in \Omega$ *there is* $C \in \mathscr{C}$ *such that* C *meets* (λ, e) *for some* $e \in E^k$, $e \neq 0$.

Remark 2.8. The proof of this result is modelled after the proof of the Rabinowitz bifurcation theorem [14, Theorem 2.3]. However, technical difficulties appear here that are not present in the aforementioned proof. In order to circumvent these difficulties, the lemmas which follow are needed. The proof then follows readily in light of [14] and is omitted. As in [14] one may verify

LEMMA 2.9. *Assume neither* (i) *nor* (ii) *of Theorem* 2.7 *hold. Then if* K, Z, *and* Γ *are as in the statement of Theorem* 2.7 *there is a bounded open set* $\mathscr{O}$ *in* $\mathbb{R}^k \times E^k$ *such that*

(a) $\bar{K} \subseteq \mathscr{O}$,

(b) $\partial\mathscr{O} \cap \mathscr{S} = = \varnothing$, *and*

(c) *if* $0 < \varepsilon < \operatorname{dist}(\partial\Gamma, Z)$, $\mathscr{O}$ *can be chosen so that* $\mathscr{O}$ *contains no trivial solutions of* (1.1)–(1.3) *other than those in an* ε *neighborhood of* Z.

LEMMA 2.10. *Suppose* (i)–(iii) *of Theorem* 2.7 *do not hold. Then if* Ω *is as in the statement of Theorem* 2.7, *there is a neighborhood* V *contained in* Ω *such that* $V \cap K_{\mathbb{R}^k} = \varnothing$, *where* $K_{\mathbb{R}^k} = \{\lambda \in \mathscr{R}^k : (\lambda, e) \in K \text{ for some } e \in E^k\}$.

Proof. Since (ii) and (iii) do not hold, there exist $\lambda \in \Omega$ such that for any subcontinuum $C \in \mathscr{C}$, $C \cap (\{\lambda\} \times E^k) = \varnothing$. Then $\lambda \notin \bar{K}_{\mathbb{R}^k}$. Otherwise, there is a sequence $\{\lambda_n\}_{n=1}^{\infty} \subseteq K_{\mathbb{R}^k}$ such that $\lambda_n \to \lambda$ in $\mathbb{R}^k$. Then there exists a sequence $\{e_n\}_{n=1}^{\infty} \subseteq E^k$, $e_n \neq 0$, such that $(\lambda_n, e_n) \in C_n$ for some subcontinuum C_n. Thus $e_n = F(\lambda_n, e_n)$. Since (i) does not hold, $\{e_n\}_{n=1}^{\infty}$ is bounded in E^k. The complete continuity of operator F implies there is a subsequence of $\{e_n\}_{n=1}^{\infty}$ (which we relabel if necessary) and an element $e \in E^k$ such that $e_n \to e$. Thus $e = F(\lambda, e)$. Hence, since $(\lambda, e) \in \overline{(Z \times \{0\}) \bigcup_{n=1}^{\infty} C_n}$, we have $\lambda \in K_{\mathbb{R}^k}$, a contradiction. Thus $\lambda \notin \bar{K}_{\mathbb{R}^k}$ and there is a neighborhood V of λ such that $V \cap K_{\mathbb{R}^k} = \varnothing$.

A modification of the construction of set $\mathscr{O}$ in Lemma 2.9 (cf. [14]) along the lines of Lemma 2.10 establishes the following result.

LEMMA 2.11. *If* (i)–(iii) *of Theorem* 2.7 *do not hold and* Ω *is as in the statement of Theorem* 2.7 *then the open set* $\mathscr{O}$ *of Lemma* 2.9 *may be chosen so that there is a neighborhood* $V' \subseteq \Omega$ *such that* $\mathscr{O} \cap (\{\lambda\} \times E^k) = \varnothing$ *for all* $\lambda \in V'$.

3. A STRONGLY COUPLED NONLINEAR STURM–LIOUVILLE SYSTEM

In this section we consider a problem from ordinary differential equations which illustrates Theorem 2.7. Let $[a, b]$ be a closed real interval and consider the system of equations

$$\begin{aligned} Lu(t) &= (\lambda + \mu)\, u(t) - 2\mu v(t) + N_1(t, \lambda, \mu, u(t), v(t)), \\ Lv(t) &= \lambda u(t) - (\lambda + \mu)\, v(t) + N_2(t, \lambda, \mu, u(t), v(t)), \end{aligned} \qquad (3.1)$$

where λ, μ are real parametrs and $t \in [a, b]$. Assume the following conditions are satisfied:

(i) $Lx \equiv -(px')' + qx$, where p is continuously differentiable and positive and q is continuous on $[a, b]$.

(ii) u and v are subject to the boundary conditions

$$\alpha x(a) + \alpha' x'(a) = 0,$$
$$\beta x(b) + \beta' x'(b) = 0,$$

where

$$(|\alpha| + |\alpha'|)(|\beta| + |\beta'|) > 0.$$

(iii) $N_i(t, \lambda, \mu, x, y)$ is continuous and also $o(|x| + |y|)$ uniformly for $t \in [a, b]$ and (λ, μ) contained in compact subsets of $\mathbb{R}^2$, for $i = 1, 2$.

Assume now that 0 is not an eigenvalue for L subject to boundary conditions (ii). Then if G is the Green's function associated with same, (3.1) is equivalent to

$$\begin{aligned} u(t) &= (\lambda + \mu) \int_a^b G(t, s)\, u(s)\, ds - 2\mu \int_a^b G(t, s)\, v(s)\, ds \\ &\quad + \int_a^b G(t, s)\, N_1(s, \lambda, \mu, u(s), v(s))\, ds, \\ v(t) &= \lambda \int_a^b G(t, s)\, u(s)\, ds - (\lambda + \mu) \int_a^b G(t, s)\, v(s)\, ds \\ &\quad + \int_a^b G(t, s)\, N_2(s, \lambda, \mu, u(s), v(s))\, ds. \end{aligned} \qquad (3.2)$$

Since the map $x \to \int_a^b G(\cdot, s)\, x(s)\, ds$ is a compact linear operator on the real Banach space E given as the subspace of those elements of $C^1[a, b]$ satisfying boundary conditions (ii), $\mathbb{R}^2 \times E^2$ is an appropriate setting in which to discuss the bifurcation phenomena associated with (3.1). In fact, (3.2) can be expressed in the form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} A & 0 \\ A & -A \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \mu \begin{bmatrix} A & -2A \\ 0 & -A \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} N_1(\lambda, \mu, u, v) \\ N_2(\lambda, \mu, u, v) \end{bmatrix}, \qquad (3.3)$$

where $A: E \to E$ is given by

$$Ax(t) = \int_a^b G(t, s)\, x(s)\, ds,$$

and $N_i: \mathbb{R}^2 \times E^2 \to E$ is given by

$$N_i(\lambda, \mu, x, y)(t) = \int_a^b G(t, s)\, N_i(s, \lambda, \mu, x(s), y(s))\, ds, \qquad i = 1, 2.$$

We note that by condition (iii), (3.3) is a special case of (1.1)–(1.3). Hence, the set $\{(\lambda, \mu) \in \mathbb{R}^2 : (\lambda, \mu, 0, 0)$ is a bifurcation point for (3.1)$\}$ is contained in the set

$$\Sigma_A = \left\{ (\lambda, \mu) \in \mathbb{R}^2 : \begin{bmatrix} u \\ v \end{bmatrix} = \left[\lambda \begin{bmatrix} A & 0 \\ A & -A \end{bmatrix} + \mu \begin{bmatrix} A & -2A \\ 0 & -A \end{bmatrix} \right] \begin{bmatrix} u \\ v \end{bmatrix} \right.$$

$$\left. \text{has a nontrivial solution } \begin{bmatrix} u \\ v \end{bmatrix} \text{ in } E^2 \right\}.$$

Note that the equation defining Σ_A is equivalent to the system

$$\begin{aligned} Lu &= (\lambda + \mu)u - 2\mu v, \\ Lv &= \lambda u - (\lambda + \mu)v, \end{aligned} \tag{3.4}$$

u and v subject to the given boundary conditions (ii). If (3.4) has a nontrivial solution $[\begin{smallmatrix} u \\ v \end{smallmatrix}]$, then u and v can be shown to solve

$$(L + \sqrt{\lambda^2 + \mu^2})(L - \sqrt{\lambda^2 + \mu^2})x = 0. \tag{3.5}$$

It is well known that L has a discrete sequence of simple eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \to +\infty.$$

It can thus be shown that $\Sigma_A = \{(\lambda, \mu): \lambda^2 + \mu^2 = \lambda_n^2,\ n = 1, 2, \ldots\}$; Σ_A is thus an infinite collection of concentric circles centered at the origin.

Theorem 2.1 guarantees that elements of Σ_A of odd algebraic multiplicity are bifurcation points. Suppose that (λ, μ) is such that $\lambda^2 + \mu^2 = \lambda_n^2$ and that there does not exist $m \neq n \in Z^+$ with $\lambda_m + \lambda_n = 0$. Let x_n be an eigenfunction for L and boundary conditions (ii) associated with λ_n. (Recall that x_n has $n - 1$ simple zeros in (a, b)). Since u and v solve (3.5), it must be the case that $u = \alpha x_n$ and $v = \beta x_n$. Then (3.4) yields

$$\begin{bmatrix} \lambda_n & 0 \\ 0 & \lambda_n \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} x_n = \begin{bmatrix} \lambda + \mu & -2\mu \\ \lambda & -(\lambda + \mu) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} x_n. \tag{3.6}$$

Then (3.6) has a nontrivial solution with $\beta = ((\lambda_n - (\lambda + \mu)/-2\mu)\alpha$ if $\mu \neq 0$, $\beta = \frac{1}{2}\alpha$ if $(\lambda, \mu) = (\lambda_n, 0)$, and $\alpha = 0$ if $(\lambda, \mu) = (-\lambda_n, 0)$. So (λ, μ) has geometric multiplicity 1.

Assume for the moment that $\lambda = \lambda_n$ and $\mu = 0$. In this case, the equation

$$\begin{bmatrix} L - (\lambda + \mu) & 2\mu \\ -\lambda & L + (\lambda + \mu) \end{bmatrix}^2 \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.7}$$

implies

$$\begin{bmatrix} L - (\lambda + \mu) & 2\mu \\ -\lambda & L + (\lambda + \mu) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \omega \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_n, \tag{3.8}$$

where $\omega \in \mathbb{R}$. Since $L + \lambda_n$ is invertible, (3.8) reduces to

$$(L - \lambda_n)u = 2\omega x_n, \qquad (L - \lambda_n)v = \omega x_n.$$

Hence $\omega = 0$. It now follows that the algebraic multiplicity at $(\lambda_n, 0)$ is 1. By the homotopy invariance of the Leray–Schauder degree, the algebraic multiplicity at (λ, μ) is odd for each (λ, μ) such that $\lambda^2 + \mu^2 = \lambda_n^2$. We have established the following result.

THEOREM 3.1. *If λ_n is an eigenvalue of L and boundary conditions* (ii) *such that $\lambda_n + \lambda_m \neq 0$ for all $m \in Z^+$, then the set $\{(\lambda, \mu): \lambda^2 + \mu^2 = \lambda_n^2\} \subseteq \mathscr{B}$.*

Theorem 2.7 is applicable to (3.1). We now consider the nontrivial solutions to (3.1) that emerge from Σ_A. We first make the following observation.

PROPOSITION 3.2. *Consider the linearization* (3.4) *of* (3.1) *and suppose that $\lambda^2 + \mu^2 = \lambda_n^2$, as in Theorem* 3.1. *Then the following hold*:

(i) *if $u = 0$ then either $v = 0$ or $(\lambda, \mu) = (-\lambda_n, 0)$.*
(ii) *if $v = 0$ then either $u = 0$ or $(\lambda, \mu) = (0, \lambda_n)$.*

THEOREM 3.3. *Suppose $(\lambda_0, \mu_0) \in \Sigma_A$, $\lambda_0^2 + \mu_0^2 = \lambda_n^2$, where λ_n is as in Theorem* 3.1. *There is a "two-dimensional"* (*in the sense of Remark* 2.6) *continuum $\mathscr{C}(\lambda_0, \mu_0)$ emerging in $\mathbb{R}^2 \times E^2$ from $(\lambda_0, \mu_0, 0, 0)$. If $(\lambda_0, \mu_0) \neq (-\lambda_n, 0)$ or $(0, \lambda_n)$ then, at least locally, all nontrivial solutions (λ, μ, u, v) in $\mathscr{C}(\lambda_0, \mu_0)$ are such that u and v have $n - 1$ simple zeros in (a, b).*

Proof. The result follows readily from Corollary 2.5 and Proposition 3.2.

There are three (not necessarily mutually exclusive) alternatives put forward in Theorem 2.7. It would be desirable to use nodal properties of solutions to (3.1) (as in [6, 14, 15, 18]) to eliminate alternative (ii) of Theorem 2.7 (i.e., continua emerging from $\{(\lambda, \mu): \lambda^2 + \mu^2 = \lambda_n^2\} \times \{(0, 0)\}$ meet $\{(\lambda, \mu): \lambda^2 + \mu^2 = \lambda_m^2\} \times \{0, 0\}$, where $m \neq n$). Unfortunately, one cannot guarantee that the zeros of a solution pair (u, v) are always simple and have the same nodal type. Thus the following theorem appears to be as strong as is possible, in general.

THEOREM 3.4. *Suppose that the nontrivial solutions (λ, μ, u, v) of* (3.1) *have the following property*: (i) *either u and v have the same number of simple zeros with no double zeros, or* (ii) *either u or v is zero. Then alternative* (ii) *of Theorem* 2.7 *does not hold.*

Proof. The Schmitt–Smith lemma [15, Theorem 2.5], suitably extended to the situation of Theorem 2.7, can be utilized to verify this result.

We conclude this section with the following simple result.

THEOREM 3.5. *Solution continua for* (3.1) *emanating from* $\{\lambda^2+\mu^2=\lambda_n^2\}\times\{(0,0)\}$, *where* λ_n *is as in Theorem* 3.1, *have unbounded* $\lambda - v$ *components provided* $N_1(t,\lambda,\mu,0,y)=0$ *and unbounded* $\mu - u$ *components if* $N_2(t,\lambda,\mu,x,0)=0$, *where* N_1 *and* N_2 *are as in assumption* (iii) *of* (3.1).

Proof. If $N_1(t,\lambda,\mu,0,y)=0$, we may apply the results of [6] to the reduced system obtained by setting $\mu=0$ and $u\equiv 0$, and analogously for $N_2(t,\lambda,\mu,x,0)=0$.

4. From a Model for the Axisymmetric Buckling of Thin Spheres

In [1], Bauer, Reiss, and Keller discuss a mathematical model for the axisymmetric buckling of hollow spheres and hemispheres. In that paper, they derive a system of nonlinear second-order differential equations. It is our purpose in this chapter to consider the bifurcation phenomena associated with a similar system in which the nonlinearity has been altered. Precisely, we consider the system

$$\begin{aligned} Ly_1+\nu y_1&=-y_2-N_1(P,K,y_1,y_2),\\ Ly_2-\nu y_2&=\frac{1-\nu^2}{K}(-Py_2+y_1+N_2(P,K,y_1,y_2)), \end{aligned}\tag{4.1}$$

where L is the differential operator given by

$$Lf=\frac{d}{dx}\left((1-x^2)\frac{df}{dx}\right)-\frac{x^2}{1-x^2}f,\tag{4.2}$$

subject to the boundary conditions

$$y_i(x)=0,\tag{4.3}$$

for $i=1,2$, $x=-1,1$. We also assume that P and K are positive parameters and that ν is a constant with value strictly between 0 and 1; N_i is to be viewed as an operator acting continuously between $\mathbb{R}^+\times\mathbb{R}^+\times(L^2[-1,1])^2$ and $L^2[-1,1]$ and satisfying $N_i(P,K,y_1,y_2)=o(\|(y_1,y_2)\|)_{(L^2[-1,1])^2}$ uniformly for (P,K) in compact subsets of $\mathbb{R}^+\times\mathbb{R}^+$.

From (4.2), Lf may be expressed as $(d/dx)((1-x^2)\ df/dx)-(1/(1-x^2))f+f$. The operator given by $f\to(d/dx)((1-x^2)\ df/dx)-(1/(1-x^2))f$ is called the associated Legendre differential operator of order

1. It is well known [8, 11, 16, 17] that this operator along with boundary conditions (4.3) is a densely defined operator on $L^2[-1,1]$ with eigenvalues $-n(n+1)$ and eigenfunctions $\rho_n=P_n^1=(1-x^2)^{1/2}\,(dP_n/dx)$, where P_n is the nth Legendre polynomial of mathematical physics; P_n^1 is called the associated Legendre polynomial of degree n and order 1. It has $n-1$ simple zeros in $(-1,1)$. Furthermore, $\{\rho_n\}_{n=1}^{\infty}$ forms a complete orthonormal basis for the real Hilbert space $\mathscr{L}^2[-1,1]$. Hence $-L$ and boundary conditions (4.3) have eigenvalues $\lambda_n=n(n+1)-1$ with corresponding eigenfunctions ρ_n. One may readily see then that the operator L subject to (4.3) has a closed extension $\bar{L}$ on $\mathscr{L}^2[-1,1]$ and $\bar{L}$ has compact inverse S on $\mathscr{L}^2[-1,1]$. Thus (4.1)–(4.3) is equivalent to

$$\begin{bmatrix} y_1\\ y_2\end{bmatrix}=\begin{bmatrix} T & 0\\ 0 & T\end{bmatrix}\begin{bmatrix} \nu & 1\\ -\dfrac{(1-\nu^2)}{K} & \dfrac{P(1-\nu^2)}{K}-\nu\end{bmatrix}\begin{bmatrix} y_1\\ y_2\end{bmatrix} + \begin{bmatrix} T & 0\\ 0 & T\end{bmatrix}\begin{bmatrix} N_1(P,K,y_1,y_2)\\ \dfrac{-(1-\nu^2)}{K}N_2(P,K,y_1,y_2)\end{bmatrix},\tag{4.4}$$

where $T=-S$. Then (4.4) is a variation of (1.1)–(1.3).

Observe now that the linearization of (4.4) is given by

$$\begin{bmatrix} y_1\\ y_2\end{bmatrix}=\begin{bmatrix} \nu T & T\\ \dfrac{-(1-\nu^2)T}{K} & -\nu T+\dfrac{P}{K}(1-\nu^2)T\end{bmatrix}\begin{bmatrix} y_1\\ y_2\end{bmatrix}.\tag{4.5}$$

Theorem 2.1 may be used to determine that elements of Σ_A which are algebraically simple are bifurcation points for (4.1)–(4.3). Equation (4.5) is equivalent to

$$\begin{aligned} -Ly_1&=\nu y_1+y_2,\\ -Ly_2&=\frac{-(1-\nu^2)}{K}y_1+\left(\frac{P(1-\nu^2)}{K}-\nu\right)y_2. \end{aligned}\tag{4.6}$$

Equation (4.6) has nontrivial solution $\left[\begin{smallmatrix} y_1\\ y_2\end{smallmatrix}\right]$ subject to (4.3) only if y_1 and y_2 satisfy

$$(-L-c+d)(-L-c-d)x=0,\tag{4.7}$$

where $c=P(1-\nu^2)/2K$ and $d=\sqrt{(\nu-(P(1-\nu^2)/2K)^2-((1-\nu^2)/K)}$. Then by (4.7) we have that $c\pm d=\lambda_n$ for some positive integer n. Hence Σ_A is the collection of rays $\{(P,K)\in\mathbb{R}^+\times\mathbb{R}^+: P=((\lambda_n+\nu)/(1-\nu^2))K+(1/(\lambda_n-\nu)),\ n=1,2,\ldots\}$. Furthermore, it can be shown that all points (P,K)

of Σ_A are algebraically simple except the countable number points, at which two rays in the collection intersect. In fact, solutions to (4.5) for such $(P, K) \in \Sigma_A$ are of the form $\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] = \omega \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] \rho_n$, where α and β are constants depending only on (P, K). It follows that "small" solutions to (4.4) inherit the nodal structure of ρ_n. Since $\mathscr{B}$ is necessarily closed, we may summarize as follows:

THEOREM 4.2. *In the setting of* (4.4), $\mathscr{B} = \Sigma_A = \{(P, K): P > 0,\ K > 0,\ P = ((\lambda_n + \nu)/(1 - \nu^2))K + (1/(\lambda_n - \nu)),\quad n = 1, 2, \ldots\}$. Furthermore, if $(P_0, K_0) \in \Sigma_A$ with $\mathrm{mult}(P_0, K_0) = 1$ and $P_0 = ((\lambda_n + \nu)/(1 - \nu^2))\, K_0 + (1/(\lambda_n - \nu))$, then locally solution pairs $\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$ for (4.4) are such that y_i has $n - 1$ simple zeros in $(-1, 1)$, $i = 1, 2$.

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